2+1-dimensional black holes with momentum and angular momentum

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Abstract

Exact solutions of Einstein's equations in 2+1-dimensional anti-de Sitter space containing any number of black holes are described. In addition to the black holes these spacetimes can possess "internal" structure. Accordingly the generic spacetime of this type depends on a large number of parameters. Half of these can be taken as mass parameters, and the rest as the conjugate (angular) momenta. The time development and horizon structure of some of these spacetimes are sketched.

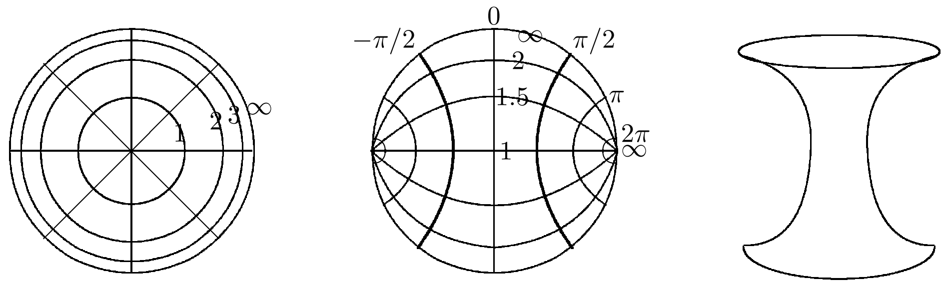
1 Introduction

The discovery [1] that 2+1-dimensional sourcefree Einstein gravity with a negative cosmological constant admits black hole spacetime was initially surprising because this theory does not admit local gravitational degrees of freedom: If the Ricci tensor is constant so is the Riemann tensor, spacetime has constant negative curvature, and is therefore locally anti-de Sitter (adS). Subsequently multi-black-hole configurations were found and classified [2, 3], but only in the time-symmetric context, when the gravitational momentum variables (extrinsic curvature) vanish. After a condensed review of these time-symmetric spacetimes in Section 2, we discuss in section 3 the more general case when the momenta do not vanish. Our conclusions are summarized in Section 4.

2 Time-symmetric multi-black-holes

2.1 Initial States

By definition, time-symmetric geometries possess a spacelike surface S such that reflection about this surface is an isometry. For multi-black-hole solutions this surface is Cauchy, so it suffices to classify the states at the moment of



(a) (b) (c)

Figure 1: Representations of initial geometries on a surface of time-symmetry for black hole spacetimes. (a) Coordinates (r, θ) on the Poincaré disk. On this scale and in the metric (1) the circles have the radii shown, in units of ℓ. (b) Coordinates of the BTZ black hole (2) on the Poincaré disk. The geodesics on which ϕ has the constant values shown (for $m = 0.25$) are represented by arcs of Euclidean circles that meet the limit circle orthogonally. The horizontal line $r/m = 1$ is likewise a geodesic. Other curves, on which r/m has the constant values shown, are equidistant from this geodesic and have constant, non-vanishing acceleration. When two copies of the heavily outlined strip, in which ϕ changes by π, are superimposed and glued together at the geodesic edges, we obtain the initial state of the BTZ metric (2), shown schematically in (c) as a surface in 3-space. This "pseudosphere" surface cannot be embedded in Euclidean space in its entirety. The space itself continues to infinite distance on both the top and bottom sheet.

time-symmetry, on the two-dimensional spacelike S whose extrinsic curvature vanishes. Because the three-dimensional spacetime has constant curvature $\Lambda < 0$ (where Λ is the cosmological constant; usually replaced by $\ell^2 = -1/\Lambda$), the instrinsic geometry of S is also one of constant negative curvature. Any such space can be put together out of pieces[1] of its universal covering space, the simply-connected two-dimensional space of constant negative curvature, H^2.

The space H^2 is conveniently represented as the *Poincaré disk*, the region $r < \ell$ of the plane with polar coordinates (r, θ) and with the metric

$$ds^2 = \frac{4}{\left(1 - \frac{r^2}{\ell^2}\right)^2} \left(dr^2 + r^2 d\theta^2\right) \tag{1}$$

The map between H^2 and the plane of polar coordinates (r, θ) is an equal-angle (conformal) map in which geodesics are represented as arcs of Euclidean circles normal to the "limit circle" $r = \ell$ (Fig. 1).

The basic time-symmetric "single" black hole is that due to Bañados, Teit-

[1] One standard construction uses a single piece, a fundamental domain of the discrete group $\mathcal{G}$ of isometries that specifies which parts of the domain's boundary are to be identified, so that $S = H^2/\mathcal{G}$. We will however use an equivalent but somewhat different description. For details see [4].

elboim and Zanelli (BTZ) [1], with metric

$$ds^2 = -\left(\frac{\rho^2}{\ell^2} - m\right) dt^2 + \left(\frac{\rho^2}{\ell^2} - m\right)^{-1} d\rho^2 + \rho^2 d\phi^2. \tag{2}$$

Putting $m = 1 = \ell$ for simplicity we find the coordinates (ρ, ϕ) of the space part of (2) to be related to the (r, θ) of (1) by

$$r^2 = \frac{\rho\cosh\phi - 1}{\rho\cosh\phi + 1} \qquad \cos\theta = \sqrt{\frac{\rho^2 - 1}{\rho^2\cosh^2\phi - 1}}\ ; \tag{3}$$

a polar coordinate plot of (3) as in Fig. 1b shows that the coordinates ρ, ϕ cover the Poincaré disk if ϕ is given an infinite range. But the coordinate ϕ of (2) is intended to have the usual range 2π of a polar angle. Fig. 1b shows in heavy outline a strip of half this size. We obtain the BTZ geometry by laying a second, identical copy of this strip on top and sewing the edges together (Fig. 1c).

Multi-black-hole initial states can be constructed in an analogous way: gluing regions together by boundary geodesics of equal length makes a smooth union since the extrinsic curvature of a 2D geodesic vanishes. Any corners of the boundary should be 90°, so that a regular neighborhood is created when four corners are glued together. Figure 2a shows an example. An identical copy is to be glued along the heavily drawn boundaries. The lightly drawn boundaries (with arrows) then become geodesic circles, and these are glued to each other. The result has the topology of a doubly punctured torus shown in Fig. 2b. Each puncture flares out to infinity, and in such a region it is isometric to an exterior region $(\rho > \ell\sqrt{m})$ of the BTZ initial state. One may describe this geometry as the initial state of two black holes that are joined through a common internal torus.

In the internal region of a general time-symmetric multi-black-hole inital state there are a number of minimal, homotopically inequivalent, closed geodesics (the curve with arrow and the dotted curves in Fig. 2b). Cutting the surface along these geodesics decomposes it into "flares" and trousers-shaped "cores." The general core geometry is obtained from two geodesic hexagons as in Fig. 2c, and it is determined by three parameters, which can be taken to be the circumferences of the trousers' legs and waist.

When assembling the general surface[2] out of cores and flares, we have to match the circumferences of the geodesics at the "seams," but we can join them with an arbitrary "twist" (a rotation along the circles). When joining a flare to a core, the twist extends to an isometry $\phi \to \phi$+const of the exterior BTZ geometry and produces no new geometry; but a twist between two cores generally changes the geometry, so at each of those seams we loose one circumference parameter and gain one twist parameter, with no net change in the total number of parameters. Thus there are three parameters for each core component. If the surface has genus g and k exteriors, there are $2g - 2 + k$ cores, and the

[2] We confine attention to orientable geometries; non-orientable ones have a double covering that is orientable.

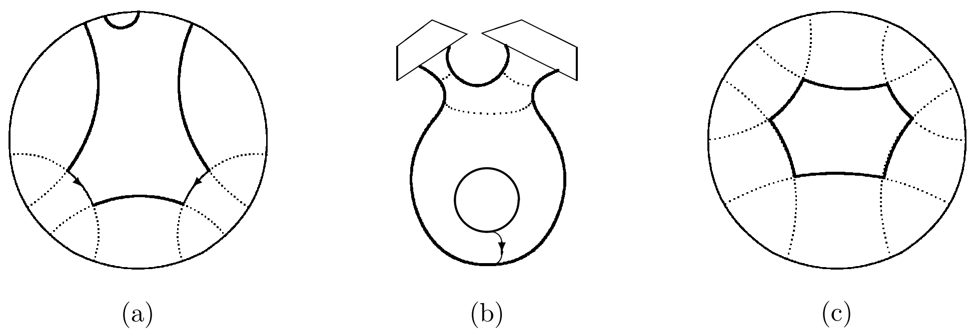


Figure 2: Construction of initial states of multi-black-hole geometries. (a) The solid black lines are geodesics in the Poincaré disk. They bound a region that has two ends at infinity. (b) Schematic representation of the geometry obtained by gluing two copies of the region in (a) along its heavily drawn boundaries, and then identifying the boundaries that have arrows. The trapezoids indicate the infinite, asymptotically adS ends. The dotted curves and the curve with the arrow are minimal geodesics that cut this figure into two semi-infinite "flares" and two internal "cores." (c) The general core is obtained by gluing two right-angle, geodesic hexagons (one of which is shown) along alternate sides.

number of parameters is $6g-6+3k$. If $g>1$ we can have $k=0$, a finite, closed "universe."

The hexagon constituents (Fig. 2c) of cores and corresponding infinite 2-gon components of flares are possible coordinate neighborhoods in which the metric can be given a standard form, such as (1). The coordinate transformations at the boundary, analogous to (3), increase in number and complexity with g and k, but are well-defined when the gluing scheme is given by a figure like Fig. 2a, and the $6g-6+3k$ parameters are specified. In this sense our figures (if labeled with the parameters) are similar to Feynman diagrams, representing well-defined mathematical expressions.

2.2 Time Development of a black hole

If we extend the metric of (2) to an infinite range of ϕ, we obtain a coordinate description of adS spacetime analogous to Rindler coordinates in Minkowski spacetime. Figure 3a shows the coordinates of (2) on the $\phi=0$ section of adS spacetime as embedded in 2+1-dimensional flat space, as well as their continuation in the usual way to $\rho<\ell\sqrt{m}$. Here the horizontal axis is spacelike and planes perpendicular to it are timelike. The numbers label the values of the coordinate t.

Expressed in new coordinates,

$$P^2=\ell^2 m-\rho^2, \qquad T=\ell\phi, \qquad \Phi=t/\ell \tag{4}$$

the metric (2) is the same expression as in the old coordinates. The new co-

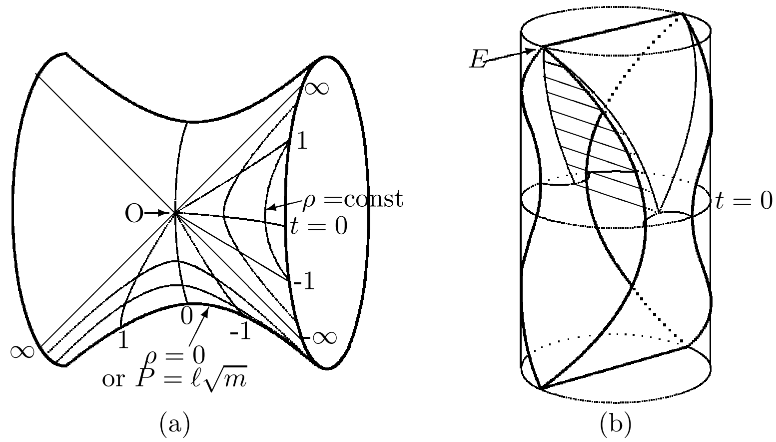


Figure 3: Time development of a single black hole without angular momentum. (a) Isometric embedding of the subspace $\phi = 0$ or the subspace $T = 0$ as a 1+1-dimensional adS space in 2+1-dimensional flat space. (b) Representation of three-dimensional adS spacetime as the interior of a cylinder in "sausage coordinates." A black hole spacetime is the double of the heavily outlined region. The striped surface is the horizon of the left front null infinity, whose endpoint is P.

ordinates interchange the role of t and ϕ in the region ρ or $P < \ell\sqrt{m}$, where these coordinates are timelike, and the spacelike surfaces are analogous to a Kantowski-Sachs universe. Therefore Fig. 3a can also be regarded as a picture of the $t = 0$ section of (2). In that case the numbers label the values of ϕ, and the initial state is the curve labeled $P = \ell\sqrt{m}$.

To return to the interpretation of (2) as a black hole, we make ϕ periodic by identifying $\phi = -\pi$ and $\phi = +\pi$. Then all P or $\rho =$ const curves are circles, with $\rho = 0$ the throat of the "wormhole" geometry, which collapses to zero size at O as the timelike ρ increases. The origin O, which previously was a coordinate singularity, becomes a non-Hausdorff singularity at $P = 0$, as in Misner space [5]. This singular line extends to infinity and marks the endpoint E of null infinity. Therefore the past of null infinity has a boundary, the horizon of the black hole.

Figure 3b is a representation adS spacetime as the interior of a cylinder in "sausage coordinates" [6]. Each horizontal slice of the cylinder is a Poincaré disk as in Fig. 1b, and the time coordinate is one in which the adS space appears static. The mantle of the cylinder represents infinity. The heavily outlined region is half ($\phi = 0$ to $\phi = \pi$, for example) of the BTZ spacetime (2). After doubling, the spacetime is no longer static, for example because the boundaries where the gluing takes place approach each other and intersect in a geodesic that ends at the point E at infinity. The heavily-outlined lozenge-shaped regions on the left front and right rear of the cylinder are the two $\mathscr{I}$s. The one in front has

endpoint at E. The horizon (striped surface) is the backward lightcone from E.

2.3 Time development of multi-black-holes

Each exterior region of a multi-black-hole initial state is isometric to a BTZ exterior, therefore the time development of each exterior will also be isometric to that shown in Fig. 3b. In particular, as seen from one such exterior, the other black holes lie behind that exterior's horizon. The whole spacetime up to the non-Hausdorff singularity can be obtained by doubling the regions of adS space corresponding to the initial neighborhoods (such as those of Fig. 2a or c). The boundaries ("seams") of these spacetime regions are generated by timelike geodesics normal to the initial boundaries.

Such two-dimensional, timelike boundaries are totally geodesic, and therefore fit together smoothly. Their intrinsic geometry is constant negative curvature (two-dimensional adS). The normal geodesics do not generate a complete surface, but only the part of a two-dimensional adS space that lies in the domain of dependence of the initial surface. Because all normal geodesics to a time-symmetric surface in adS spacetime intersect in one point, all the seams also intersect in one point T. When they are analytically extended to complete surfaces they intersect along spacelike geodesics, which will form the non-Hausdorff singularity after gluing. Thus the top (and bottom) of the region to be doubled looks like a pyramid-shaped "tent" whose ridge lines are the singularities (Fig. 4a). In the interior the ridge lines come together at the point T. From there they run to infinity, where they define the endpoints of each exterior's $\mathscr{I}$. The horizon is obtained by running a lightcone backwards from each of these endpoints to the points of intersection with another such backward lightcone. (For details of this construction see [3].)

3 Angular momentum

The general BTZ metric describes a "single" black hole (with two asymptotically adS regions) that has angular momentum J in addition to mass m. As we will see below, the metric with $J \neq 0$ can be obtained from the time-symmetric one, which has $J = 0$, by changing the rules by which its two halves are glued together. To fix ideas we first consider such rule change within the time-symmetric class.

3.1 Alternative ways of gluing

Consider a three-black-hole initial state, obtained by gluing together two copies of the region between three disjoint geodesics on the Poincaré disk. Previously we have identified each point of the heavily drawn curves in the upper disk of Fig. 4b with the one vertically below it on the lower disk. The geodesic's neighborhood is invariant under "translation" isometries that move each point on the geodesic by a constant distance. Therefore we get an equally smooth

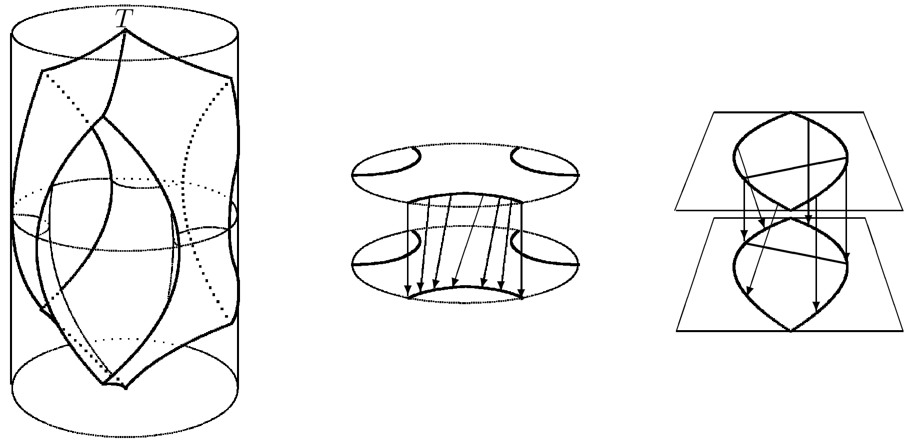


Figure 4: (a) The time development in sausage coordinates of a region of adS spacetime that becomes a three-black-hole when doubled. (b) An alternative way of gluing two regions of Poincaré disks, seen in perspective, to obtain a three-black-hole initial state. (c) An alternative way of gluing two vertical (timelike) slices of the "sausage" of Fig. 3.

surface if we move the points on the lower disk by such an isometry, so that the gluing identifies points that are connected by the arrows in Fig. 4b. The result of gluing with a shift depends on the amount of shift, for example because the size of the minimal closed geodesics around the adjacent black holes, and hence their masses, depend on it. However, a change in mass is all that can happen to the initial state, because we know that all time-symmetric three-black-hole initial states are characterized by just three mass parameters. The shift can always be "transformed away" by changing the geodesic seams that are to be glued together.[3]

In the space-time picture of a black hole (Fig. 3b) or of multi-black-holes (Fig. 4a) the seams are timelike hypersurfaces of constant negative curvature. These surfaces are invariant under a 3-parameter group of isometries. Again we can consider different gluing rules, depending on what isometry is applied at the seam. We want to distinguish those ways of re-gluing time-symmetric black holes that lead to new types of spacetimes.

We have already seen that we do not get a new class of spacetimes from re-gluing a seam by an isometry that leaves the surface of time-symmetry invariant. Similarly, if we re-glue a time-symmetric BTZ black hole by a time translation, we again get a spacetime with a surface of time-symmetry, that is, another time-symmetric black hole. This is illustrated in Fig. 4c in the timelike subspace

[3]The same circumstance in flat space is illustrated by gluing a cylinder out of a piece of paper either with or without such a shift. In either case one gets a cylinder. If there is no shift, the seam is parallel to the cylinder's axis. If there is a shift, the cylinder's radius is smaller, and the seam is a helix on the cylinder.

obtained by slicing Fig. 3b with a vertical plane from left back to right front. Two copies of this plane are shown in perspective. The two pairs of curves on the planes are the seams, at $\phi = 0$ and π on the top plane, and $\phi = \pi$ and 2π on the bottom plane. The arrows connect points that are to be glued together.[4] The heavy lines on the planes connect smoothly to form a closed geodesic[5] in the new surface of time-symmetry produced by this re-gluing. (In the three-dimensional picture the new surface of time-symmetry is obtained from the old one (labeled $t = 0$ in Fig. 3b) by a "Lorentz transformation" isometry that has the geodesic $t = 0$, $\phi = \pi$ as an axis.)

In order to obtain a new class of black holes, with angular momentum, we re-glue a time-symmetric black hole by an isometry in the seam that has a fixed point at $t = 0$.

3.2 BTZ black hole with angular momentum

In the metric (2) for the static BTZ black hole, introduce new coordinates T, φ, R,

$$\begin{aligned} t &= T + \left(\frac{J}{2m}\right)\varphi \\ \phi &= \varphi + \left(\frac{J}{2m\ell^2}\right)T \\ R^2 &= \rho^2\left(1 - \frac{J^2}{4m^2\ell^2}\right) + \frac{J^2}{4m} \end{aligned} \tag{5}$$

where $J < 2m\ell$ is a constant with dimension of length, and define another new constant

$$M = m + \frac{J^2}{4m\ell^2}. \tag{6}$$

In terms of these new quantities the metric (2) becomes

$$ds^2 = -N^2 dT^2 + N^{-2} dR^2 + R^2\left(d\varphi + \frac{J}{2R^2}dT\right)^2 \tag{7}$$

where

$$N^2 = \left(\frac{R}{\ell}\right)^2 - M + \left(\frac{J}{2R}\right)^2. \tag{8}$$

[4] This alternative to gluing along vertical arrows can also be illustrated in Fig. 3a, where the numbers at the bottom are values of ϕ: cut the figure into two halves by a vertical plane through the center and perpendicular to the picture plane, rotate one half against the other about a horizontal axis, and re-glue.

[5] In an accurate plot of sausage coordinates these geodesic segments would not look straight as they do in this qualitative picture.

Equation (7) is the metric for a black hole with angular momentum J. In this metric, the new coordinate φ is taken as periodic. When it changes by its period, 2π, the old coordinates of (2) change by

$$t \to t + \frac{\pi J}{m} \qquad \phi \to \phi + 2\pi \tag{9}$$

This tells us that in order to obtain a black hole with angular momentum, by re-gluing a time-symmetric one, we should apply a "boost" by $\pi J/m$ at the $\phi = 2\pi$ seam about the (old) horizon.

The construction by re-gluing a $J = 0$ black hole spacetime does not yield the full $J \neq 0$ spacetime, because the pieces that we glue together end at the ridge lines ($r = 0$) that become non-Hausdorff singularities in the $J = 0$ case. When $J \neq 0$ the spacetime can be extended beyond the ridge line, to $R = 0$, which would correspond to negative r^2. Otherwise stated, we do not obtain a representation of the full spacetime when we cut it into two pieces, because the two cuts we make along the seams intersect each other when we get too far from the initial surface. Nevertheless, a piece of the spacetime is enough to characterize it, and we can use it to deduce the number of parameters needed.

3.3 Multi-black-holes with angular momenta

Our purpose is to characterize spacetimes that have angular momenta and are counterparts to the time-symmetric multi-black-hole spacetimes of section 2. The basic building block is the three-black-hole spacetime of Fig. 4a. It has three totally geodesic seams, each of which is a 1+1-dimensional adS spacetime, of constant negative curvature Λ, exactly the same as the seams the single black hole of section 3.2. Each of these seams can therefore be re-glued smoothly by applying an adS isometry. These isometries form a three-parameter group. As before, only a one-dimensional subset leads to spacetimes without any surface of time-symmetry, so there is one effective parameter per seam. The general three-black-hole state is therefore characterized by three configuration parameters, which can be taken to be the three masses, and three momentum parameters, the boost angles at the three seams.

The actual angular momentum of any one of the black holes (and therefore also its actual mass M, equation (6)) depends on the boost parameters of its two adjacent seams, and on the fixed point of the boost. Note that a boost at a seam is an isometry only within that seam (and in a neighborhood of the seam), but it cannot in general be extended to the whole spacetime. Once we have a three-black-hole spacetime with angular momentum, we can forget about the time-symmetric geometry that was used to construct it. The core will extend only to each leg's local (outer) horizon at R_{H} (where R_{rmH} is the larger root of $N^2(R_{\mathrm{H}}) = 0$). The geometry is locally reflection-symmetric about R_{H}, and it can be matched there to other cores with the same local geometry.

The general time-symmetric multi-black-hole spacetime considered here was put together out of three-black-hole cores and and exteriors (Fig. 2b). We can put a spacetime together in topologically the same way if the cores have angular

momenta. We only have to match masses and angular momenta at the seams, because the neighborhood of each seam (including the entire interior region, $R < R_{\mathrm{H}}$) depends only on that seam's M and J. So, when matching two cores we lose two of the parameters characterizing the separate cores. In the time-symmetric case we also gained one parameter, which we called a twist because it describes a rotation along the seam. In the case of space-time, the neighborhood of a seam is the same as a single black hole metric (7). It is therefore locally invariant no only under a change in ϕ ("twist") but also under a change in T "boost". Re-identifying an internal seam with a twist and a boost gives us two additional parameters back. Thus our general multi-black-hole is characterized by twice the number of parameters necessary to specify a time-symmetric one.

4 Conclusions

We have seen that sourceless 2+1-dimensional Einstein theory with a negative cosmological constant admits solutions with all spatial two-dimensional topologies that can carry a constant negative curvature metric.[6] Among these are multi-black-hole spacetimes that have several asymptotically anti-de Sitter regions, each of which is characterized by a mass and an angular momentum. A number of additional parameters are needed (except for the three-black-hole configuration) to characterize the internal structure. Half of these are configuration parameters that specify internal sizes or angular relationships; the other half are momentum parameters, which vanish if the space-time is time-symmetric.

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[6] In addition, the torus topology is also admitted, but not as a time-symmetric state. For example, identifying t and ϕ periodically for $\rho < \ell\sqrt{m}$ in (2) yields the torus topology.

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